Ricci Collineations of the Bianchi Type-II, VIII, and IX Spacetimes

İ. Yavuz† and U. Camcı†

† Çanakkale Onsekiz Mart University, Art and Science Faculty, Department of Mathematics, 17100 Çanakkale, Turkey
e-mail : iyavuz@comu.edu.tr,
e-mail : ucamci@hotmail.com

Abstract

Ricci and contracted Ricci collineations of the Bianchi type II, VIII, and IX space-times, associated with the vector fields of the form (i) one component of $\xi^a(x^b)$ is different from zero and (ii) two components of $\xi^a(x^b)$ are different from zero, for $a, b = 1, 2, 3, 4$, are presented. In subcase (i.b), which is $\xi^a = (0, \xi^2(x^a), 0, 0)$, some known solutions are found, and in subcase (i.d), which is $\xi^a = (0, 0, 0, \xi^4(x^a))$, choosing $S(t) = const. \times R(t)$, the Bianchi type II, VIII, and IX space-times is reduced to the Robertson-Walker metric.

1 Introduction

In addition to isometries that leave invariant the metric tensor under Lie transport and provide information of the symmetries inherent in space-time, spacetimes may admit other symmetries that do not leave the metric tensor invariant. Whereas the geometrical nature of a space-time is encoded into the metric tensor, by virtue of the Einstein equations, the physics is given more explicitly by the Ricci tensor. Besides the space-time metric, the curvature and Ricci tensors are other important candidates which play the significant role in understanding the geometric structure of space-times in general relativity. These general symmetry considerations (e.g., Ricci collineations, curvature collineations), based on the invariance properties of different geometric objects under Lie transport, have been classified by Katzin et al.[1]

The locally rotationally symmetric (LRS) metric for the spatially homogeneous Bianchi type-II($\delta = 0$), VIII($\delta = -1$), and IX($\delta = +1$) cosmological models can generally be written in the form[2, 3]

$$ds^2 = dt^2 - S^2(dx - hdz)^2 - R^2(dy^2 + f^2dz^2) \tag{1}$$

where S and R are arbitrary functions of t only and

$$f(y) = \begin{pmatrix} y \\ siny \\ sinhy \end{pmatrix}, h(y) = \begin{pmatrix} -y^2/2 \\ cosy \\ -coshy \end{pmatrix} \;\; for \;\; \delta = -\frac{f''}{f} = \begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}$$

(prime denotes differentiation with respect to y).

We give the geometric classification of 3-parameter Lie groups for the Bianchi type II, VIII, and IX space-times in the following table (from Ref.4). In this table, the last column gives a possible realisation of the algebra in terms of translations, rotations, boosts, null rotations and a dilation (scaling) acting on subspaces of Minkowski space, where a and b are the parameters which occur in the generators ($a = 0$ for Bianchi type V, $a = b$ for Bianchi type IV).

Table I. Geometric classification of 3-parameter Lie groups for the Bianchi type II, VIII, and IX space-times.

Group type	Isomorphic to	(Sub)group of	Generators
II	G_1, H_1	homothetic motions of null 2-plane	$\partial_y, \partial_x, a\partial_y + b(x\partial_x + y\partial_y)$
VIII	$SO(2,1)$	2-pseudosphere	$t\partial_x + x\partial_t, t\partial_y + y\partial_t, x\partial_y - y\partial_x$
IX	$SO(3)$	2-sphere	$y\partial_x - x\partial_y, z\partial_y - y\partial_z, x\partial_z - z\partial_x$

The Killing equation can be written in component form as

$$(K_{ab}) : g_{ab,c}k^c + g_{ac}k^c_{,b} + g_{cb}k^c_{,a} = 0, \tag{2}$$

where the comma represents partial differentiation with respect to the respective coordinate, k^a are the components of the KV field, and Latin indices take values $1, 2, 3, 4$. As for all Bianchi space-times, the Bianchi type II, VIII, and IX space-times considered here also admit Killing Vectors (KVs), or isometries [5] and homothetic motions [6], i.e. symmetries that scale all distances by the same constant factor and preserve the null geodesic affine parameters.

According to the classification of Katzin et al. [1], a given Riemannian space-time will admit Ricci collineations (RCs) provided one can find a solution of the equations [7, 8]

$$\pounds_\xi R_{ab} = \xi^c \nabla_c R_{ab} + R_{ac} \nabla_b \xi^c + R_{cb} \nabla_a \xi^c = 0, \tag{3}$$

where $\pounds_\xi$ denotes the operation of Lie differentiation with respect to the vector ξ^a and ∇ represents covariant derivative operator. Clearly, if a solution ξ^a to equation (3) exists (which corresponds to an infinitesimal point mapping $x^a \rightarrow x^a + \epsilon\xi^a$), then it represents symmetry property of the particular Riemannian spacetime, and this symmetry property will correspond to at least a G_1 group of Ricci collineation. In a torsion free space in a coordinate basis, the RC equation (3) represented by (C_{ab}) reduces to a simple partial differential equation (pde)[9]

$$(C_{ab}) : R_{ab,c}\xi^c + R_{ac}\xi^c_{,b} + R_{cb}\xi^c_{,a} = 0, \tag{4}$$

where ξ^a are the components of the RC vector. Also, a less restrictive class of symmetries corresponds to the Family of Contracted Ricci Collineation(FCRC), defined by

$$g_{ab}\pounds_\xi R_{ab} = 0. \tag{5}$$

Each member of the FCRC symmetry mapping satisfies $g^{ab}\pounds_\xi(T_{ab} - \frac{1}{2}Tg_{ab}) = 0$, which leads to the following conservation law generator for a space-time admitting RC [8]:

$$\nabla_a[(-g)^{\frac{1}{2}}(T^a_b - \frac{1}{2}T\delta^a_b)\xi^b] = \nabla_a[(-g)^{\frac{1}{2}}R^a_b\xi^b] = \partial_a[(-g)^{\frac{1}{2}}R^a_b\xi^b] = 0.$$

The relationship between the RCs and the isometries have been discussed in detail by Amir, Bokhari, and Qadir [10]. They provide the complete classification of the RCs according to the nature of the Ricci tensor which is constructed from a general spherically symmetric and static metric.

If $h = 0$ and $f = 1$, then the metric (1) coincide the LRS Bianchi type I space-time [11] while if we take $h = 0, f = 1$ and $R = S$ in the metric (1), then we have the Robertson-Walker spacetime with $k = 0$. Green et al.[12] and Nuñez et al.[8] have provided an example of RC and FCRC symmetries of Robertson-Walker spacetime, and they have confined their study to symmetries generated by the vector fields of the form, respectively,

$$\xi = (0, 0, 0, \xi^4(r, \theta, \phi, t)), \tag{6}$$

and

$$\xi = (\xi^1(r, t), 0, 0, \xi^4(r, t)). \tag{7}$$

In this paper we investigate some symmetry properties of the Bianchi type II, VIII, and IX space-times by considering RCs associated with the following vector fields
(i) one components of $\xi^a(x^b)$ is different from zero:

$$\begin{aligned}
(i.a)\ \ \xi^a &= (\xi^1(x^a), 0, 0, 0),\\
(i.b)\ \ \xi^a &= (0, \xi^2(x^a), 0, 0),\\
(i.c)\ \ \xi^a &= (0, 0, \xi^3(x^a), 0),\\
(i.d)\ \ \xi^a &= (0, 0, 0, \xi^4(x^a)),
\end{aligned}$$

(ii) two components of $\xi^a(x^b)$ is different from zero. In this case, we consider the following subcases only.

$$\begin{aligned}
(ii.a)\ \ \xi^a &= (\xi^1(x^a), \xi^2(x^a), 0, 0),\\
(ii.b)\ \ \xi^a &= (0, \xi^2(x^a), \xi^3(x^a), 0),\\
(ii.c)\ \ \xi^a &= (0, \xi^2(x^a), 0, \xi^4(x^a)),
\end{aligned}$$

where $x^a = (x, y, z, t)$. Furthermore, for these vector fields, we obtain a FCRC symmetry family. The motivation to study this kind of symmetries comes from the fact that, to the best of our knowledge, a systematic analysis of RCs and FCRC associated to Bianchi universes is not available in the literature.

In section 2, we study in detail RCs for the vector fields in the cases (i) and (ii) and also, in section 3, we are dealt with FCRC.

2 Ricci Collineations

The nonvanishing components of the Ricci tensor, corresponding to the metric (1), read

$$\begin{aligned}
R_{11} &= S^2A(t),\ \ R_{13} = -h[R_{11} + \beta S^2/(2R)^2],\ \ R_{22} = R^2B(t),\\
R_{33} &= h^2[R_{11} + \beta S^2/R^2] + f^2R_{22},\ \ R_{44} = (\frac{2\ddot{R}}{R} + \frac{\ddot{S}}{S})
\end{aligned} \tag{8}$$

where a dot indicates derivative with respect to time and we have defined

$$A(t) = \frac{\ddot{S}}{S} + 2\frac{\dot{R}\dot{S}}{RS} + \alpha\frac{S^2}{2R^4},\ \ B(t) = \frac{\ddot{R}}{R} + \frac{\dot{R}\dot{S}}{RS} + \left(\frac{\dot{R}}{R}\right)^2 - \left[\alpha\frac{S^2}{2R^4} - \frac{\delta}{R^2}\right] \tag{9}$$

such that α and β are

$$\beta = \frac{f'h'}{f\,h} - \frac{h''}{h}, \quad \alpha = \left(\frac{h'}{f}\right)^2.$$

For the Bianchi type-II, VIII, and IX spacetime, we obtain that $\alpha = 1$ and $\beta = 0$.

For the metric (1), the RC equations (4), generated by an arbitrary vector field $\xi^a(x, y, z, t)$, reads

$$(C_{11}) : R_{11,4}\xi^4 + 2R_{11}\xi^1_{,1} + 2R_{13}\xi^3_{,1} = 0,$$

$$(C_{22}) : R_{22,4}\xi^4 + 2R_{22}\xi^2_{,2} = 0,$$

$$(C_{33}) : R_{33,2}\xi^2 + R_{33,4}\xi^4 + 2R_{13}\xi^1_{,3} + 2R_{33}\xi^3_{,3} = 0,$$

$$(C_{44}) : R_{44,4}\xi^4 + 2R_{44}\xi^4_{,4} = 0,$$

$$(C_{12}) : R_{11}\xi^1_{,2} + R_{22}\xi^2_{,1} + R_{13}\xi^3_{,2} = 0,$$

$$(C_{13}) : R_{13,2}\xi^2 + R_{13,4}\xi^4 + R_{11}\xi^1_{,3} + R_{33}\xi^3_{,1} + R_{13}(\xi^1_{,1} + \xi^3_{,3}) = 0,$$

$$(C_{14}) : R_{11}\xi^1_{,4} + R_{44}\xi^4_{,1} + R_{13}\xi^3_{,4} = 0,$$

$$(C_{23}) : R_{22}\xi^2_{,3} + R_{33}\xi^3_{,2} + R_{13}\xi^1_{,2} = 0,$$

$$(C_{24}) : R_{22}\xi^2_{,4} + R_{44}\xi^4_{,2} = 0,$$

$$(C_{34}) : R_{33}\xi^3_{,4} + R_{44}\xi^4_{,3} + R_{13}\xi^1_{,4} = 0.$$

In the above equations we will consider two different cases :

Case (i): One component of $\xi^a(x^b)$ is different from zero.
In this case, there are four subcases:

Subcases (i.a)-(i.c). From the RC equations, for these three subcases, we obtain that ξ^1, ξ^2 and ξ^3 are constants. It means that these three RC vectors $\xi^i, i = 1, 2, 3$ represent a *translation* along x, y and z directions, respectively. Further, from subcase (i.b) we get

$$R_{11} = R_{22} = 0 \quad \Leftrightarrow \quad A(t) = B(t) = 0. \tag{10}$$

Therefore, using (8), (9) and (10), we find that

$$\frac{\ddot{R}}{R} + \frac{\ddot{S}}{S} + 3\frac{\dot{R}\dot{S}}{RS} + (\frac{\dot{R}}{R})^2 + \frac{\delta}{R^2} \quad = \quad 0,$$

which leads to

$$[R(RS)^{\dot{}}\,]^{\dot{}} = -\delta S. \tag{11}$$

When we use the transformations of the time-coordinate $d\bar{t} = Sdt$ and $dt = SR^2 d\tau$ in (11), we obtain the general solutions for Bianchi type II($\delta = 0$) metric, respectively,

$$(RS)^2 = c_1\bar{t} + c_2, \quad and \quad (RS)^2 = e^{2(c_3\tau + c_4)}, \tag{12}$$

where c_1, c_2, c_3, and c_4 are integration constants. These solutions are found by Lorenz [13]. Also, making the scale transformation $d\bar{t} = Sdt$ into (11), we get the general solution for Bianchi type II, VIII, and IX metrics,

$$(RS)^2 = -\delta(\bar{t})^2 + c_5\bar{t} + c_6, \tag{13}$$

where c_5, c_6 are constants. This corresponds to the solution found by Lorenz [14].

subcase (i.d). From the RC equations, we get

$(C_{11}) : R_{11,4}\xi^4 = 0, \;\; (C_{22}) : R_{22,4}\xi^4 = 0, \;\; (C_{33}) : R_{33,4}\xi^4 = 0,$

$(C_{44}) : R_{44,4}\xi^4 + 2R_{44}\xi^4_{,4} = 0,$

$(C_{13}) : R_{13,4}\xi^4 = 0, \;\; (C_{14}) : R_{44}\xi^4_{,1} = 0,$

$(C_{24}) : R_{44}\xi^4_{,2} = 0, \;\; (C_{34}) : R_{44}\xi^4_{,3} = 0.$

In this case, from (C_{11}) and (C_{22}), we find that

$$\dot{R}_{11} = 0 \quad \Leftrightarrow \quad R_{11} = a_1, \tag{14}$$

$$\dot{R}_{22} = 0 \quad \Leftrightarrow \quad R_{22} = a_2, \tag{15}$$

where a_1, a_2 are constants. It is clear that Eqs. (C_{33}) and (C_{13}) are identically satisfied. Further, from $(C_{14}), (C_{24})$ and (C_{34}), we have $\xi^4 = \xi^4(t)$. Now, using (C_{44}), we obtain that

$$\xi^4(t) = \frac{c}{\sqrt{\mid R_{44} \mid}}, \tag{16}$$

where c is a constant. Choosing $S(t) = a_3 R(t), a_3 = \text{const.}$, from (14) and (15) we obtain

$$\frac{\ddot{R}}{R} + 2\left(\frac{\dot{R}}{R}\right)^2 + \frac{2k}{R^2} = 0, \tag{17}$$

where k is a disposable constant, and can be set to ± 1 or 0. Then, the solution for eq. (17) represents the Robertson-Walker spacetime in general. The RCs of this space-time for the RC vector (7) and the solutions for eq. (17) have been found by Nuñez et al. [8]

Case (ii). The two components of $\xi^a(x^b)$ are different from zero. In this case, we will consider the following subcases only.

Subcase (ii.a). From the RC equations, we obtain that

$(C_{33}) : R_{33,2}\xi^2 + 2R_{13}\xi^1_{,3} = 0,$

$(C_{12}) : R_{11}\xi^1_{,2} + R_{22}\xi^2_{,1} = 0,$

$(C_{13}) : R_{11}\xi^1_{,3} + R_{13,2}\xi^2 = 0,$

$(C_{23}) : R_{13}\xi^1_{,2} + R_{22}\xi^2_{,3} = 0,$

$(C_{11}) : R_{11}\xi^1_{,1} = 0, \ \ (C_{22}) : R_{22}\xi^2_{,2} = 0,$

$(C_{14}) : R_{11}\xi^1_{,4} = 0, \;\; (C_{24}) : R_{22}\xi^2_{,4} = 0.$

From the above equations, using $(C_{11}), (C_{22}), (C_{14})$ and (C_{24}), we find that $\xi^1 = \xi^1(y, z)$ and $\xi^2 = \xi^2(x, z)$. As a result of (C_{13}) and (C_{23}), we obtain that ξ^2 vanishes, and therefore ξ^1 becomes a constant. Then it follows that there is a *translation* along the x-direction, since ξ^2 vanishes and ξ^1 is a constant.

Subcase (ii.b). For this subcase, RC equations yield

$(C_{33}) : R_{33,2}\xi^2 + 2R_{33}\xi^3_{,3} = 0,$

$(C_{12}) : R_{22}\xi^2_{,1} + R_{13}\xi^3_{,1} = 0,$

$(C_{13}) : R_{33}\xi^3_{,1} + R_{13}\xi^3_{,3} = 0,$

$(C_{23}) : R_{22}\xi^2_{,3} + R_{33}\xi^3_{,2} = 0,$

$(C_{11}) : 2R_{13}\xi^3_{,1} = 0, \;\; (C_{22}) : 2R_{22}\xi^2_{,2} = 0,$

$(C_{14}) : R_{13}\xi^3_{,4} = 0, \;\; (C_{24}) : R_{22}\xi^2_{,4} = 0.$

Using (C_{22}) and (C_{24}), we have $\xi^2 = \xi^2(x, z)$, while eqs. (C_{11}) and (C_{14}) give $\xi^3 = \xi^3(x, z)$. Therefore, eq. (C_{13}) becomes

$$(C_{13}) : R_{13}\xi^3_{,3} + R_{13,2}\xi^2 = 0.$$

From this equation, obviously, we find

$$\xi^2(x, z) = -\frac{h}{h'}\xi^3_{,3}. \tag{18}$$

Then, substituting eq. (18) into (C_{33}), we obtain that ξ^3 is a function of y only. Therefore, from (18), we find that ξ^2 vanishes and also, from (C_{12}) and (C_{23}), that ξ^3 is a constant. In this case, since ξ^2 vanishes and ξ^3 is a constant, this means that there is a *translation* along the z-direction.

Subcase (ii.c). In this subcase, we have the following RC equations:

$(C_{11}) : R_{11,4}\xi^4 = 0,$

$(C_{22}) : R_{22,4}\xi^4 + 2R_{22}\xi^2_{,2} = 0,$

$(C_{33}) : R_{33,2}\xi^2 + R_{33,4}\xi^4 = 0,$

$(C_{44}) : R_{44,4}\xi^4 + 2R_{44}\xi^4_{,4} = 0,$

$(C_{13}) : R_{13,2}\xi^2 + R_{13,4}\xi^4 = 0,$

$(C_{24}) : R_{22}\xi^2_{,4} + R_{44}\xi^4_{,2} = 0,$

$(C_{12}) : R_{22}\xi^2_{,1} = 0, \quad (C_{14}) : R_{44}\xi^4_{,1} = 0,$

$(C_{23}) : R_{22}\xi^2_{,3} = 0, \quad (C_{34}) : R_{44}\xi^4_{,3} = 0,$

Using eqs. $(C_{12}), (C_{14}, (C_{23})$ and (C_{34}), we have that ξ^2 and ξ^4 are functions of y and t only. From (C_{11}), we find that

$$\dot{R}_{11} \quad = \quad 0 \quad \Leftrightarrow \quad R_{11} = const. = a_1$$

and by using (C_{13}) we see that ξ^2 vanishes. Therefore, eqs. (C_{22}) and (C_{33}) give

$$\dot{R}_{22} = 0 \quad \Leftrightarrow \quad R_{22} = const. = a_2$$

On the other hand, the component ξ^4 is determined by (C_{44}), and since from (C_{24}) that $\xi^4 = \xi^4(t)$, we see that it keeps the form (16). Therefore, this subcase is reduced to the subcase (i.d).

3 Family of Contracted Ricci Collineations

The Family of Contracted Ricci collineations (FCRC) for the Bianchi type II, VIII, and IX metric (1) takes the form

$$\begin{aligned} A(t)\xi^1_{,1} + B(t)\left(\frac{f'}{f}\xi^2 + \xi^2_{,2}\right) + h[B(t) - A(t)]\xi^3_{,1} & \\ +B(t)\xi^3_{,3} + R_{44}\left[\left(\frac{\dot{R}_{44}}{R_{44}} + 2\frac{\dot{R}}{R} + \frac{\dot{S}}{S}\right)\xi^4 + 2\xi^4_{,4}\right] &= 0. \end{aligned} \qquad (19)$$

It is not possible to find a solution to (19) without imposing some additional restrictions either on the metric or on the vector field ξ^a.

In subcase (i.a), we find from (19) that $\xi^1_{,1} = 0$, i.e. $\xi^1 = \xi^1(y, z, t)$ for $B(t) \neq 0$. A first example of proper (nondegenerate) FCRC vector is obtained by setting subcase (i.b) in the above equation. In the present case, eq.(19) becomes

$$B(t)\left(\frac{f'}{f}\xi^2 + \xi^2_{,2}\right) = 0.$$

This equation can be integrated by demanding that $B(t)$ be different from zero. Thus we find

$$\xi^2 = \frac{k_1(x, z, t)}{f(y)},$$

where $k_1(x, z, t)$ is an arbitrary function of integration with respect to the coordinate y and $f(y) = y, siny$, or $sinhy$ for the Bianchi types II, VIII, and IX respectively. By inserting subcase (i.d) in (19), we get

$$\left(\frac{\dot{R}_{44}}{R_{44}} + 2\frac{\dot{R}}{R} + \frac{\dot{S}}{S}\right)\xi^4 + 2\xi^4_{,4} = 0.$$

If we solve this equation, we can easily obtain

$$\xi^4 = \left(\frac{k_2(x, y, z)}{R^2 S R_{44}}\right)^{1/2}$$

where $k_2(x, y, t)$ is arbitrary function of the spatial variables.

4 Conclusion

RC equation (4) for metric (1) represents ten nonlinear pde's for seven unknowns (four $\xi^a, a = 1, 2, 3, 4$, which are functions of all the space-time coordinates, three functions of t given by R_{11}, R_{22}, and R_{44}, since $R_{13} = hR_{11}$ and $R_{33} = h^2R_{11} + f^2R_{22}$). In the present paper, the possible value of the RC classification of Bianchi type II, VIII, and IX space-times for the vector fields (i) and (ii) are given.

In section 2, we worked out a classification of the RCs for the RC vector of the form (i) and (ii). In the subcases (i.a), (i.b) and (i.c), we find from the RC equations (C_{ab}) that the components ξ^1, ξ^2 and ξ^3 are constants for the Bianchi type II, VIII, and IX space-time (1). Therefore, it follows that these three subcases represent *translation* along x, y and z directions, respectively. Further, when we choose RC vector ξ^a, as for subcases (ii.a), (ii.b) and (ii.c), then ξ^2 always vanishes and the other two components (i.e., ξ^1 and ξ^3) become a constant, but ξ^4 is a function of t only. Therefore, the subcases (ii.a), (ii.b) and (ii.c) are reduced to the subcases (i.a), (i.c) and (i.d), respectively. Consequently, it should be noted that (i.d) and (ii.c) types of symmetry vector ξ^a would lead to proper RC for this cosmological model; the other cases are degenerate (i.e. nonproper). Notice that every KV is an RC vector for the LRS Bianchi type II, VIII, and IX space-times, but the converse is not true. We remark that solutions (12) and (13) which were found by Lorenz [13, 14] for metric (1) have been obtained by us in subcase (i.b). Also, in subcase (i.d), our metric (1) is reduced to the Robertson-Walker space-time as a special case if we choose $S(t) = const. \times R(t)$. Finally, in section 3, for the subcases (i.b) and (i.d), we find some proper FCRC vector.

The RC equations (C_{ab}) have been abtained by assuming that $A(t), B(t)$ [i.e.,R_{11}, R_{13}, R_{22}, and R_{33}], and R_{44} do not vanish. Nevetheless there may exist solutions that correspond to the vanishing of any of the above quantities. This case will be discussed in another paper.

References

[1] G. H. Katzin, J. Levine, and H. R. Davis, *J. Math. Phys.* **10**, 617-629 (1969).

[2] A. K. Banerjee, and N. O. Santos,*Gen.Rel.Grav.* **16**, 217-224, (1984).

[3] T. Sing, and A. K. Agrawal,*Astrophys. Space Sci.* **191**, 61-88 (1991).

[4] S. T. C. Siklos, *Astrophysics and Relativistic Cosmology, (Proc. XIVth GIFT International Seminar on Theoretical Physics, Spain)*, X. Fustero and E. Verdaguer, eds. (World Scientific, Singapore,1983), pp. 218.

[5] D. Kramer, H. Stephani. E. Herlt, and M. A. H. MacCallum, *Exact Solution of Einstein Field Equations* (Cambridge U.P., Cambridge, 1980).

[6] A. K. Banerjee, S. B. Duttachoudry, and A. K. Sanyal *Gen. Rel. Grav.***18**, 461 (1986).

[7] W. R. Davis, L. H. Green, and L. K. Norris, *Nuovo Cimento* **B34**, 256-280 (1976).

[8] L. Nuñez, U. Percoco, and V. M. Villalba,*J. Math. Phys.* **30**, 137-139 (1990).

[9] A.H. Bokhari, and A.Qadir,*J. Math. Phys.***34**, 3543-3552 (1993).

[10] M. J. Amir, A. H. Bokhari, and A. Qadir,*J. Math. Phys.* **35**, 3005-3012 (1994).

[11] M. A. H. MacCallum, *In General Relativity: An Einstein Centenary Survey*, S. W. Hawking and W. Israel, eds. (Cambridge U.P., Cambridge, 1979), p.533.

[12] L. H. Green, L. K. Norris, D. R. Oliver Jr., and W. R. Davis, *Gen. Rel. Grav.***8**, 731-736 (1977).

[13] D. Lorenz,*Phys.Let.***A79**, 19-20 (1980).

[14] D. Lorenz,*Phys.Rev.***D22**, 1848-1852 (1980).